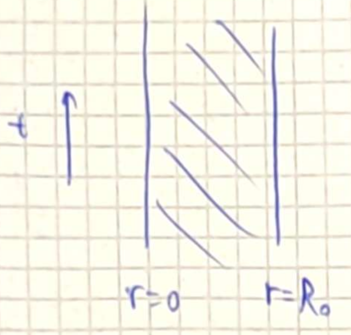
General Relativity | Week 8

First non-trivial solution of the vacuum Einstein equation:

Schwarzschild (Dec 1915): Exterior of a spherically symmetric star:



$r > R_0$: vacuum

$$g_M = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\, \underbrace{g_{S^2}}_{d\theta^2+\sin^2\theta\, d\phi^2}$$

$M > 0$: "Schwarzschild mass"

Effect on the motion of "far away", free-falling observers:

Like moving in gravitational field (Newtonian) of star of mass M.

The above metric: Can be considered independently of the star.

Well-defined on the manifold $\mathcal{M} = \mathbb{R}_t \times (2M, +\infty)_r \times \mathbb{S}^2_{\theta,\phi}$.

Properties:
- Spherically symmetric: The action of $SO(3)$ on the $\mathbb{S}^2$ factor by isometries.
- Static: ∂_t is Killing and hypersurface orthogonal
- Asymptotically flat: $g_M = \eta + O(r^{-1})$ as $r \to +\infty$

(So has conformal boundary $\mathcal{I}^{\pm}$ as Minkowski)

Killing algebra: 4 dimensional (translations in t & rotations).

What happens at $r = 2M$? Is it a singularity?

- In the case of the simpler metric $g = -t^2dt^2 + dx^2$: $t=0$ looks like a singularity, but if we set $\tilde{t} = \frac{1}{2}t^2$, then $g = -d\tilde{t}^2 + dx^2$.

Non-smooth change of coordinates!

Similarly in the usual (θ, ϕ) coordinates on $\mathbb{S}^2$: the round metric looks singular at the poles $\theta = 0, \pi$, but this is merely a coordinate singularity.

An intrinsic approach to check whether $r = 2M$ is a "true" singularity (as opposed to a "coordinate" one): Look for coordinate systems constructed "geometrically", such as the ones constructed using null geodesics (we might still encounter coordinate singularities, e.g. caustics)

(Note: In the Riemannian setting: The question of smooth extendibility can be easily addressed, working for instance with the Riemannian distance function).

Let $\gamma(s) = (t(s), r(s), \theta(s), \phi(s))$ be a null geodesic in $(\mathcal{M}, g_M)$

As we will see in the exercises, $s \mapsto (\theta(s), \phi(s))$ moves along a great circle (you can deduce this by noting that $\langle \dot\gamma, \Omega \rangle = \text{const}$ for any spherical rotation v.f Ω)

So: Without loss of generality: $\theta = \frac{\pi}{2} = \text{const}$

Then, conserved quantities:
- $\langle \dot\gamma, \dot\gamma \rangle = 0$
- $\langle \dot\gamma, \partial_t \rangle = -E$ (energy)
- $\langle \dot\gamma, \partial_\phi \rangle = L$ (angular momentum).

Enough conserved quantities to reduce to a 1^{st} order system:

$E = (1 - \frac{2M}{r}) \dot t$, $L = r^2 \dot\phi$ so

$$0 = \langle \dot\gamma, \dot\gamma \rangle = -\frac{E^2}{1-\frac{2M}{r}} + \frac{\dot r^2}{1-\frac{2M}{r}} + \frac{L^2}{r^2} \Rightarrow \boxed{\dot r^2 = E^2 - \frac{1-\frac{2M}{r}}{r^2} L^2}$$

If $L = 0$ (radial null geodesic), and $\dot r(0) < 0$, $E > 0$ (future directed, ingoing)

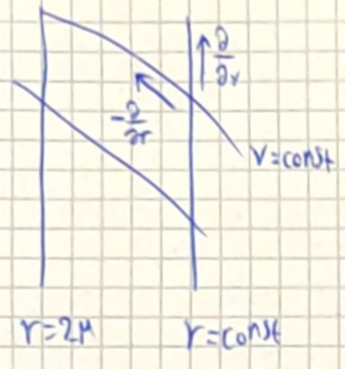
Then $\dot r = \text{const} = -E$ so we reach $r = 2M$ in <u>finite affine time</u>

If $r^* = r + 2M\log(r-2M)$ $\left(\text{so } \frac{dr^*}{dr} = \frac{1}{1-\frac{2M}{r}}\right)$

and $v = t + r^*$:

Then along the ingoing radial geodesics: $\frac{dv}{ds} = \dot{t} + \frac{dr^*}{dr}\dot{r} = 0$

(so $v = \text{const}$ are ingoing null hypersurfaces) (v: "advanced time")



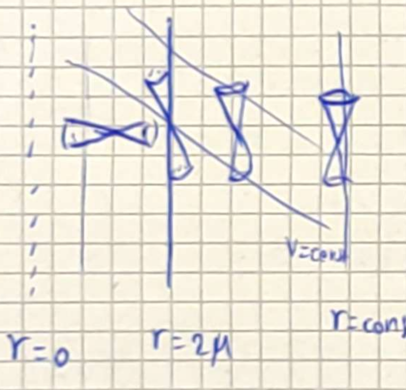
In (v, r, θ, ϕ) coordinates

(Eddington - Finkelstein):

$$g = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Smooth across $r = 2M$!

- Note: The vector field $\frac{\partial}{\partial r}$ depends on the whole coordinate system.
- In this system: $\frac{\partial}{\partial v}$ is the static Killing field.



Extension: From $\mathcal{M} = \mathbb{R}_v \times (2M, +\infty)_r \times \mathbb{S}^2_{\theta,\phi}$ to

$\tilde{\mathcal{M}} = \mathbb{R}_v \times (0, +\infty)_r \times \mathbb{S}^2_{\theta,\phi}$

$\{r = 2M\}$: Null hypersurface.

Note: - If $r_0 < 2M$: The $\{r = r_0\}$ hypersurfaces are spacelike

(r becomes a time function in the region $r < 2M$)

- If we start from $r < 2M$: No future directed causal curve can escape $\Longrightarrow$ Black hole!
- Ingoing radial null geodesics "hit" $r = 0$ in finite affine time

This is a true singularity: Kretschmann scalar $\mathcal{K} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{C}{r^6}$

$\to \infty$ at $r = 0$

(Note that K is coordinate independent!)

- Sbierski: No C^0 extension beyond $r=0$.

What is the <u>maximal</u> extension we can construct?

<u>Kruskal - Szekeres:</u>

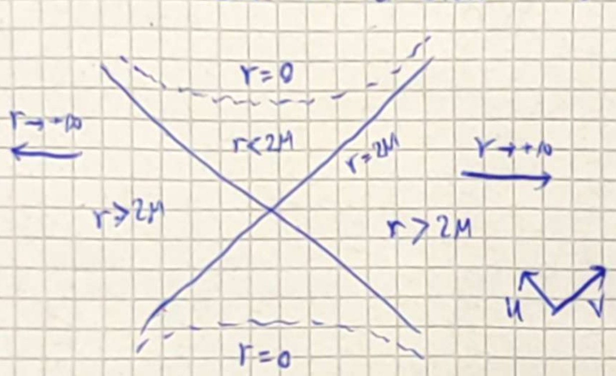
Double null coordinates:

(u,v), defined implicitly by $\frac{v}{u} = -e^{-t/2M}$, $v\cdot u = (1-\frac{r}{2M})\, e^{r/2M} = -\frac{1}{2M} e^{r^*/2M}$

$$\left(\text{as } r\to+\infty:\ \begin{aligned} \log(-u) &= \frac{t+r^*}{4M} - \frac{1}{2}\log(2M) \\ \log(v) &= \frac{-t+r^*}{4M} - \frac{1}{2}\log(2M) \end{aligned}\right)$$

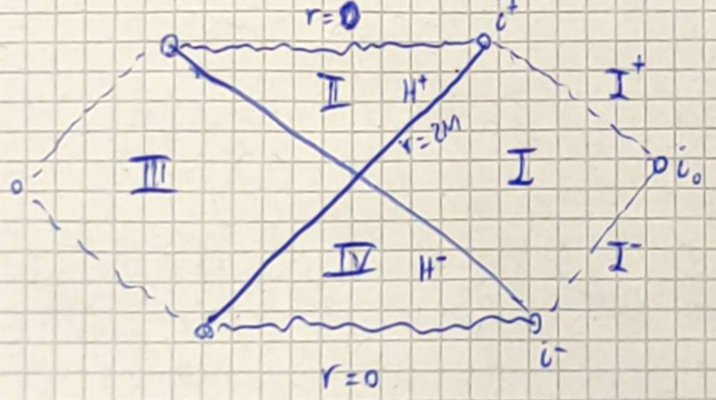
Then $g_M = -32\frac{M^3}{r}\exp(-\frac{r}{2M})\, du\, dv + r^2 g_{S^2}$

Can be extended to the region $\boxed{u\cdot v < 1}$



Setting $\tilde{u}=\tilde{u}(u)$ and $\tilde{v}=\tilde{v}(v)$ to compactify:

Penrose diagram:



- Trapped null geodesics: Go to i^+
- Note: The image of a causal curve on the penrose diagram is causal with respect to the background metric.

I: Original region, II: Black hole region,

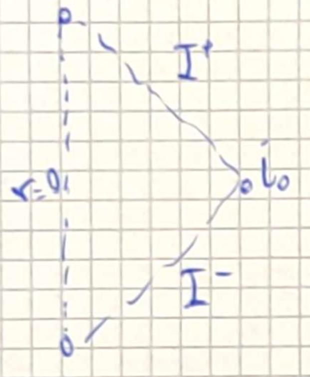
III: Symmetric to I, IV: White hole region

The maximally extended Schwarzschild:

- Globally hyperbolic, Cauchy hypersurface $\simeq \mathbb{R} \times \mathbb{S}^2$
- Asymptotically flat with 2 as. flat ends
- The function r: has geometric meaning, $r(p) = \sqrt{\frac{\text{Area}(\text{Orbit}_p)}{4\pi}}$, where Orbit_p is the orbit of p under the $SO(3)$ action.

<u>Exercise</u>: The region I & II is also globally hyperbolic. Can you find a Cauchy hypersurface? Is it complete as a Riemannian manifold?

The expression of the metric also makes sense for $M < 0$ (unphysical)



- "Naked" singularity at $r=0$: Visible from infinity.
- Not globally hyperbolic

M has the meaning of mass: For a massive (timelike) observer:

$\langle \dot\gamma, \dot\gamma \rangle = -\mu < 0$

Geodesic equation: $\dot r^2 = E^2 - \frac{1-\frac{2M}{r}}{r^2} L^2 - \left(1 - \frac{2M}{r}\right)\mu$

For $r \gg 2M$: Trajectory similar to Newtonian orbit for gravitational potential $\Phi = -\frac{M}{r}$

<u>Birkhoff's theorem</u>: <u>Every</u> spherically symmetric (M^{3+1}, g) solution to the vacuum Einstein equations $Ric = 0$ is locally isometric to (M_{Sch}, g_M) for some $M \in \mathbb{R}$.

- No degrees of freedom for the vacuum equations in sph. symmetry

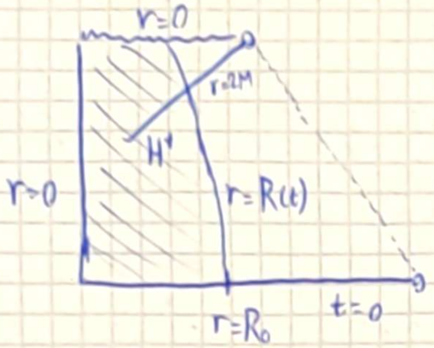
Schwarzschild black hole: "eternal" & 2-ended.

It looks unphysical as a solution. However, black holes can form dynamically

Oppenheimer - Snyder (1939)

For a sph. symmetric, homogeneous ball of dust: ($\leftarrow$ pressureless fluid)

($T_{\mu\nu} = \rho \cdot u_\mu u_\nu$)



By Birkhoff's theorem: The vacuum exterior has to be Schwarzschild.

Reissner - Nordström black hole:

Spherically symmetric / static : Solution of Einstein - Maxwell system

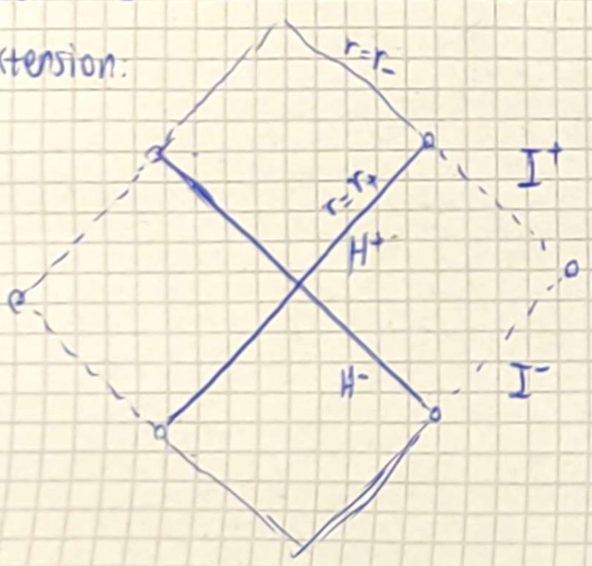
$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}[F] \\ \nabla^\mu F_{\mu\nu} = 0 \\ \nabla_{[\alpha} F_{\beta\gamma]} = 0 \end{cases}$$

$$T_{\mu\nu}[F] = F_\mu{}^\lambda F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

Exterior metric:

$$g = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 g_{S^2}$$

When $0 < |Q| < M$ (subextremal case): Maximal globally hyperbolic extension:



$r_\pm$: Roots of $r^2 - 2Mr + Q^2 = 0$

At $r = r_-$: The metric extends smoothly (Cauchy horizon).

But every extension is necessarily not globally hyperbolic.

We want to study more general solutions of the Einstein Vacuum equations $Ric = 0$

Written in coordinates:

$$\underbrace{-\frac{1}{2}\Box_g g_{\mu\nu} + \nabla_{(\mu}\Gamma_{\nu)}}_{\text{principal symbol}} + g\cdot\partial g\cdot\partial g = 0 \qquad ①$$

$$\Gamma_\nu = g_{\mu\nu} g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} \sim \partial g$$

$$\Box_g f = \frac{1}{\sqrt{-\det g}}\partial_\alpha\left(\sqrt{-\det g}\, g^{\alpha\beta}\partial_\beta f\right) = div(df) \qquad \text{(Laplace - Beltrami operator)}$$

Note: $\Box_g$ is of hyperbolic signature:

When $g = \eta$: $\Box_\eta = -\partial_t^2 + \partial_1^2 + \dots + \partial_n^2$

(classical wave operator)

① is not an equation of some well-known type (e.g elliptic, hyperbolic)

(for good reason: If it was, we wouldn't have the freedom of changing coordinates)

In wave coordinates: $\Box_g x^\mu = 0 \iff g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0 \iff \Gamma_\nu = 0$

$$① \rightsquigarrow \begin{cases} \Box_g g_{\alpha\beta} = Q_{\alpha\beta}(g, \partial g) & \longleftarrow \text{Quasilinear wave eqn!} \\ \Gamma_\nu = 0 & \longleftarrow \text{Overdetermined!} \end{cases}$$

(this will restrict our freedom to choose initial data)